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ISOMETRIC IMMERSIONS IN SYMMETRIC SPACES

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1. Introduction

Let M and $\bar{M}$ be complete Riemannian manifolds (we make this assumption throughout the paper). Denote the sectional curvatures of M and $\bar{M}$ by K and $\bar{K}$, respectively. R. Hermann [4] has proved the following theorem.

Theorem A. *Suppose that $\bar{M}$ is simply connected, M is isometrically immersed as a closed submanifold of $\bar{M}$, and $K \leq \bar{K} \leq 0$. Then $H_i(M) = 0$ for $i > \dim \bar{M} - \dim M$. (Here the homology groups have coefficients in an arbitrary field.)*

It follows from Theorem A that M cannot be immersed in $\bar{M}$ if $\dim \bar{M} < 2 \dim M$ and M is compact. In fact E. Stiel [10] has proved the following result.

Theorem B. *Suppose M is compact and $\bar{M}$ is simply connected. If $\bar{K}$ is constant, $\bar{K} \leq 0$, $K \leq 0$ and $\dim \bar{M} < 2 \dim M$, then M cannot be isometrically immersed in $\bar{M}$.*

In this paper we show that the hypotheses of Theorem A can be weakened provided $\bar{M}$ is a symmetric space. At the same time we obtain a generalization of Stiel's theorem. An example of our results is the following.

Theorem (3.4). *Suppose M is isometrically immersed as a closed submanifold of $\bar{M}$, $\bar{M}$ is a simply connected symmetric space with nonpositive curvature, and $\sup K \leq \min \bar{K} - \max \bar{K}$. Then M has the homotopy type of a CW-complex with no cells of dimension greater than $\dim \bar{M} - \dim M$.*

We also give variations of the above theorem under the hypotheses that M is a minimal variety of $\bar{M}$ or that $\bar{M}$ is not simply connected. Furthermore we consider the case when $\bar{M}$ has positive curvature, and generalize some results of Ôtsuki [9].

2. The Hessian of the distance function

In this section we assume that M is isometrically immersed as a closed C^∞ submanifold of $\bar{M}$. Let M_p and $\bar{M}_p$ be the tangent spaces of M and $\bar{M}$ at a point $p \in M$, and write $\bar{M}_p = M_p \oplus M_p^\perp$. We denote by $\langle , \rangle$ the metric tensor of M or $\bar{M}$, and by $R_{xy}(x, y \in M_p)$ and $\bar{R}_{zw}(z, w \in \bar{M}_p)$ the curvature operators of

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M and $\overline{M}$, respectively. If $\|x \wedge y\| \neq 0 \neq \|z \wedge w\|$, we write $K_{xy} = \|x \wedge y\|^{-2}\langle R_{xy}x, y\rangle$ and $\overline{K}_{zw} = \|z \wedge w\|^{-2}\langle \overline{R}_{zw}z, w\rangle$ for the sectional curvatures of M and $\overline{M}$, respectively. Let T be the configuration tensor of M in $\overline{M}$. Here for $x \in M_p$, $T_x \colon \overline{M}_p \to \overline{M}_p$ is a skew symmetric linear operator such that $T_x x$ is the acceleration in $\overline{M}$ at p of a geodesic in M starting at p and having initial velocity x. T contains the same information as the second fundamental form $S_z (z \in M_p^{\perp})$, which is related to T by the formula $S_z x = T_x z$. (See [3] for further details.)

Let ρ denote the distance function of $\overline{M}$. We assume the existence of a point $m_0 \in \overline{M}$, $m_0 \notin M$, such that M is disjoint from the cut locus of m_0. (Such a point always exists if $\overline{M}$ has nonpositive curvature and is simply connected.) Define a real-valued function f on M by $f(m) = \rho(m, m_0)$; then f is differentiable. We determine the Hessian H_f at a critical point $m \in M$ of f.

There exists a unique unit speed geodesic $\sigma\colon [0, b] \to \overline{M}$ from m_0 to m. We denote by σ' the velocity of σ and by Z' the covariant derivative of a vector field Z along σ. There are no points on σ conjugate to m_0. For $z, w \in \overline{M}_m$ let Z and W be the unique Jacobi vector fields along σ (i.e., $Z'' = \overline{R}_{Z\sigma'}\sigma'$) such that $Z(0) = W(0) = 0$ and $Z(b) = z$, $W(b) = w$. Define

$$
\begin{aligned}
Q(z, w) &= \int_0^b \{\langle Z', W'\rangle - \langle \overline{R}_{\sigma' Z}\sigma', W\rangle\}(t)dt \\
&= \langle Z', W\rangle(b) .
\end{aligned}
$$

Then [5, p. 219] the Hessian H_f is given by the formula

$$H_f(x, y) = Q(x, y) + \langle T_x y, z\rangle \qquad \text{for } x, y \in M_m ,$$

where $z = \sigma'(b) \in M_m^{\perp}$.

If $\overline{M}$ is a symmetric space, we can explicitly determine the quadratic form Q.

Lemma (2.1). *Suppose $\overline{M}$ is a symmetric space.*

(i) *Then the eigenvectors of $w \to \overline{R}_{zw}z (w \in \overline{M}_m)$ diagonalize Q.*

(ii) *If w is an eigenvector of $w \to \overline{R}_{zw}z$ corresponding to the eigenvalue $\lambda = \overline{K}_{wz}$, then*

$$
Q(w, w) = \begin{cases} \|w\|^2\sqrt{\lambda}\cot(b\sqrt{\lambda}) , & \text{if } \lambda > 0 , \\ \|w\|^2 b^{-1} , & \text{if } \lambda = 0 , \\ \|w\|^2\sqrt{-\lambda}\coth(b\sqrt{-\lambda}) , & \text{if } \lambda < 0 , \end{cases}
$$

where $b = \rho(m, m_0)$.

Proof. Let $\lambda_0, \cdots, \lambda_l$ be the eigenvalues of $w \to \overline{R}_{zw}z$, and $z_0, \cdots, z_l$ be the corresponding eigenvectors. We take $z_0 = z$ (so that $\lambda_0 = 0$). Since $w \to \overline{R}_{zw}z$ is a symmetric linear operator, each λ_i is real and we may assume that $\{z_0, \cdots, z_l\}$ forms an orthonormal basis of $\overline{M}_m$. Let $P_0, \cdots, P_l$ be parallel vector fields on σ such that $P_i(b) = z_i$ $(i = 0, \cdots, l)$. Then, because $\bar{M}$ is a symmetric space, we have

$$\bar{R}_{\sigma'(t)P_i(t)}\sigma'(t) = \lambda_i P_i(t)$$

$(i = 0, \cdots, l)$ for $0 \leq t \leq b$. If Z_i is a Jacobi vector field on σ with $Z_i(0) = 0$ and $Z_i(b) = z_i \in \bar{M}_m$, we may write

$$Z_i = f_i P_i \ ;$$

then $f_i'' + \lambda_i f_i = 0$, $f_i(b) = 1$, and $f_i(0) = 0$ for $i = 0, \cdots, l$. It is not hard to see that $Q(z_i, z_j) = 0$ for $i \neq j$. Then

$$Q(z_i, z_j) = \langle Z_i', Z_j \rangle(b) = f_i'(b)\delta_{ij} \ .$$

Since each f_i is trigonometric, linear, or hyperbolic, the lemma follows.

3. Immersions in symmetric spaces of nonpositive curvature

We begin by sharpening a result of Hermann [4] in the case that $\bar{M}$ is a symmetric space of rank 1.

Theorem (3.1). *Suppose that $\bar{M}$ is a simply connected symmetric space with $\bar{K} \leq 0$. Let M be an immersed closed submanifold of dimension n. Suppose k is an integer such that for all $p \in M$ and all $z \in M_p^\perp$ with $\| z \| = 1$ at least k of the eigenvalues κ of S_z (counted according to multiplicity) satisfy*

$$\kappa \leq \sqrt{-\max \bar{K}} \ . \tag{3.1}$$

Then M has the homotopy type of a CW-complex with no cells of dimension greater than $n - k$.

Proof. Choose $m_0 \in \bar{M}$, $m_0 \notin M$, so that the function $f\colon M \to R$ defined in § 2 has no degenerate critical points. This is possible by Sard's theorem (see [1, p. 225]). The Hessian H_f at a critical point $m \in M$ of f is given by

$$H_f(x, y) = Q(x, y) - \langle S_z(x), y \rangle \ .$$

If x is an eigenvector corresponding to an eigenvalue κ of S_z satisfying (3.1), then

$$H_f(x, x) \geq \| x \|^2 (\sqrt{-\max \bar{K}} - \kappa) \geq 0 \ .$$

It follows that any subspace of M_m on which H_f is negative definite must have dimension less than or equal to $n - k$. Since f is obviously bounded from below, Theorem (3.1) now follows from [5, Theorem 3.5].

We next investigate some sufficient conditions that the hypotheses of Theorem (3.1) be satisfied. The following lemma is originally due to Ôtsuki [8], but we use it in the form given by O'Neill [7].

Lemma (3.2). *Let V^d and V^e be two vector spaces over R of dimensions d and e respectively, let $\langle\ ,\ \rangle$ denote a positive definite bilinear form on either V^d or V^e, and suppose $\varphi\colon V^d \times V^d \to V^e$ is a symmetric bilinear function. Define*

$$\Delta(x, y) = \frac{\langle \varphi(x, x), \varphi(y, y)\rangle - \|\varphi(x, y)\|^2}{\|x \wedge y\|^2},$$

provided $x, y \in V^d$ are linearly independent. Let $x \in V^d$ be a point on the unit sphere of V^d at which $x \to \|\varphi(x, x)\|$ assumes its minimun. Then

(i) $\langle x, y\rangle = 0$, *if* $\varphi(x, y) = 0$ *and* $\varphi(x, x) \neq 0$;

(ii) $\langle \varphi(x, y), \varphi(x, x)\rangle = 0$, *if* $\langle x, y\rangle = 0$;

(iii) *if* $\langle x, y\rangle = 0$ *and* $\|y\| = 1$, *we have*

$$\Delta(x, y) + 3\|\varphi(x, y)\|^2 \geq \|\varphi(x, x)\|^2 . \tag{3.2}$$

We shall need the following consequence of Lemma (3.2).

Lemma (3.3). *Suppose that the hypotheses of Lemma* (3.2) *are satisfied; in addition, assume that there is a $\Delta \geq 0$ such that $\Delta(x, y) \leq \Delta$ whenever x and y are linearly independent, and that $e < d$. Let $k = d - e$; then for each unit vector $z \in V^e$, there exist $x_1, \cdots, x_k \in V^d$ with $\|x_i\| = 1$, $\langle x_i, x_j\rangle = 0$ $(i \neq j)$, such that $\langle \varphi(x_i, x_i), z\rangle^2 \leq \Delta$ for $i, j = 1, \cdots, k$.*

Proof. Let x_1 be a point on the unit sphere of V^d at which $x \to \langle \varphi(x, x), z\rangle^2$ achieves its minimum. If $\varphi(x_1, x_1) \neq 0$, then since $e < d$, there exists $y \in V^d$ such that $\|y\| = 1$, $\varphi(x_1, y) = 0$, and $\langle x_1, y\rangle = 0$. From (3.2) we conclude that $\langle \varphi(x_1, x_1), z\rangle^2 \leq \Delta$.

Let W be the orthogonal complement of x_1. Evidently if $e < d - 1$, the above argument applies to the restriction of φ to W. We conclude that there exists $x_2 \in W$ with $\|x_2\| = 1$ such that $\langle \varphi(x_2, x_2), z\rangle^2 \leq \Delta$. We continue this process until $x_1, \cdots, x_k$ are constructed.

Next we use Theorem (3.1) and Lemma (3.3) to obtain the theorem mentioned in the introduction.

Theorem (3.4). *Suppose that $\overline{M}$ is a simply connected symmetric space with* $\max \overline{K} \leq 0$. *Let M be an immersed closed submanifold. If* $\sup K \leq \min \overline{K} - \max \overline{K}$, *then M has the homotopy type of a CW-complex with no cells of dimension greater than* $\dim \overline{M} - \dim M$.

Proof. In Lemma (3.3) we take φ to be the restriction of T to a map from $M_p \times M_p$ into $M_p^{\perp}$. Then by the Gauss equation [3], $\Delta(x, y) = K_{xy} - \overline{K}_{xy}$ whenever $x, y \in M_p$ are linearly independent. We take $\Delta = \sup K - \min \overline{K}$ in Lemma (3.3). It follows that at least $2 \dim M - \dim \overline{M}$ of the eigenvalues

κ of S_z (where $z \in M_p^{\perp}$, $\|z\| = 1$) satisfy

$$\kappa^2 \leq \sup K - \min \bar{K} \leq - \max \bar{K} .$$

Now Theorem (3.4) follows from Theorem (3.1).

The following corollary is an immediate consequence of Theorem (3.4).

Corollary (3.5). *Suppose that $\bar{M}$ is a complete simply connected symmetric space with* $\max \bar{K} \leq 0$, *and that M is a compact Riemannian manifold such that* $\max K \leq \min \bar{K} - \max \bar{K}$. *If* $\dim \bar{M} < 2 \dim M$, *then M cannot be isometrically immersed in $\bar{M}$.*

In the case where $\max \bar{K} = 0$, Corollary (3.5) is a special case of a result of [6]. If $\max K > \min \bar{K} - \max \bar{K}$, we can still obtain a lower bound on the diameter $\bar{d}(M)$ in $\bar{M}$ of a compact immersed submanifold M.

Theorem (3.6). *Suppose that $\bar{M}$ is a simply connected symmetric space (of rank 1) with* $\max \bar{K} < 0$, *and that M is a compact Riemannian manifold with* $\dim \bar{M} < 2 \dim M$ *and* $\max K > \min \bar{K} - \max \bar{K}$. *If M is isometrically immersed in $\bar{M}$, then*

$$\bar{d}(M) \geq \frac{1}{\sqrt{-\min \bar{K}}} \coth^{-1} \left(\frac{\max K - \min \bar{K}}{-\max \bar{K}} \right)^{1/2} . \tag{3.3}$$

Proof. We take $m_0 \in M$, and let $m \in M$ be a point at which f assumes its maximum. (Then m is possibly a degenerate critical point of f, and f is possibly non-differentiable at m_0, but this does not matter.) We have $H_f(x, x) \leq 0$ for all $x \in M_m$. Let $b = f(m)$; then $b \leq \bar{d}(M)$. It follows from Lemma (2.1) that

$$\| T_x x \| \geq Q(x, x) \geq \sqrt{-\max \bar{K}} \coth (\bar{d}(M) \sqrt{-\min \bar{K}}) \tag{3.4}$$

for all $x \in M_m$ with $\|x\| = 1$. In particular, (3.4) holds if we choose x to be a point on the unit sphere of M_m at which $\|T_x x\|$ achieves its minimum. Thus we have by Lemma (3.2) that

$$\max K \geq \min \bar{K} - \max \bar{K} \coth^2 (\bar{d}(M) \sqrt{-\min \bar{K}}) .$$

This last inequality is equivalent to (3.3).

In the case that $\bar{M}$ is not simply connected, we can still say something about the topology of immersed compact submanifolds

Theorem (3.7). *Suppose that $\bar{M}$ is a complete locally symmetric space with* $\max \bar{K} \leq 0$, *and that M is a compact Riemannian manifold isometrically immersed in $\bar{M}$ with* $\dim \bar{M} < 2 \dim M$. *If* $\max K \leq \min \bar{K} - \max \bar{K}$, *then $\pi_1(M)$ is infinite.*

Proof. If $\pi_1(M)$ were finite, then the inverse image of M in the universal covering space of $\bar{M}$ would have compact components.

The *mean curvature vector field* H of an isometrically immersed submanifold M of a Riemannian manifold $\bar{M}$ is defined by

$$H = \sum_{i=1}^{n} T_{E_i} E_i ,$$

where $n = \dim M$, and $\{E_1, \cdots, E_n\}$ is any orthonormal frame field on an open subset of M. M is a *minimal variety* of $\bar{M}$ if and only if $H \equiv 0$. We state without proof the following theorem which is similar to the theorems proved above.

Theorem (3.8). *Suppose that $\bar{M}$ is a complete locally symmetric space with* $\max \bar{K} \leq 0$, *and that M is a compact Riemannian manifold isometrically immersed in $\bar{M}$ with* $\dim \bar{M} < 2 \dim M$. *Write* $n = \dim M$.

(i) *If* $\max \| H \| \leq n\sqrt{-\max \bar{K}}$, *then $\pi_1(M)$ is infinite.*
(ii) *If $\bar{M}$ is simply connected, then*

$$\bar{d}(M) \geq \frac{1}{\sqrt{-\min \bar{K}}} \coth^{-1}\left(\frac{\max \| H \|}{n\sqrt{-\max \bar{K}}}\right),$$

provided $\max \bar{K} < 0$.

4. Immersions in compact symmetric spaces

In this section we prove some results analogous to those in § 3. However, the theorems in this section are weaker because in the case of nonnegative curvature the symmetric bilinear form Q may not be positive semidefinite.

If $\bar{M}$ is any Riemannian manifold and $q \in \bar{M}$, we denote by $U(q, b)$ the closed "geodesic" neighborhood consisting of all points whose distance to q is less than or equal to b. Now suppose that $\bar{M}$ is a compact simply connected symmetric space, so that $\min \bar{K} \geq 0$ and $\max \bar{K} > 0$. Let M be an immersed compact submanifold of dimension n. If $q \in \bar{M}$ and $0 < b \leq \frac{\pi}{2} (\max \bar{K})^{-1/2}$,

define $k(q, b)$ to be the greatest integer k such that for all $p \in M \cap U(q, b)$ and all $z \in M_p^{\perp}$ with $\| z \| = 1$, at least k of the eigenvalues κ of S_z (counted according to multiplicity) satisfy

$$\kappa \leq \sqrt{\min \bar{K}} \cot (b\sqrt{\max \bar{K}}) . \tag{4.1}$$

Theorem (4.1). *Suppose that $\bar{M}$ is a compact simply connected symmetric space, and let M be an immersed compact submanifold of dimension n. If* $0 < b \leq \frac{\pi}{2} (\max \bar{K})^{-1/2}$, *then*

(i) *for almost all $q \in \bar{M}$, $M \cap U(q, b)$ has the homotopy type of a CW-complex with no cells of dimension greater than $n - k(q, b)$;*

(ii) *for all* $q \in \bar{M}$ *the conditions* $k(q, b) > 0$ *and* $0 < b' < b$ *imply that* M *is not contained in* $U(q, b')$.

Proof. According to Crittenden [2], the cut locus of $\bar{M}$ coincides with the conjugate locus of $\bar{M}$. Hence we may apply the results of § 2. The rest of the proof is along the same lines as that of Theorems (3.1) and (3.6), and so we omit it.

Similarly, the following theorem can be proved in the same fashion as Theorem (3.4).

Theorem (4.2). *Suppose that* $\bar{M}$ *is a compact simply connected symmetric space. Let* M *be an immersed compact submanifold with*

$$\max K \leq \min \bar{K} \csc^2 (b\sqrt{\max \bar{K}}) , \tag{4.2}$$

where $0 < b \leq \frac{\pi}{2} (\max \bar{K})^{-1/2}$. *Then*

(i) *for almost all* $q \in \bar{M}$, $M \cap U(q, b)$ *has the homotopy type of a CW-complex with no cells of dimension greater than* $\dim \bar{M} - \dim M$;

(ii) *for all* $q \in \bar{M}, M$ *is not contained in* $U(q, b')$, *provided* $\dim \bar{M} < 2 \dim M$ *and* $0 < b' < b$.

We remark that sufficient conditions that (4.1) and (4.2) hold are that $\kappa \leq 0$ and $\max K \leq \min \bar{K}$, respectively.

If $\max K > \min \bar{K}$, we can obtain a lower bound on the diameter $\bar{d}(M)$ in $\bar{M}$ of M, just as we did in Theorem (3.6).

Theorem (4.3). *Suppose that* $\bar{M}$ *is a compact simply connected symmetric space, and let* M *be a compact Riemannian manifold with* $\max K > \min \bar{K}$ *and* $\dim \bar{M} < 2 \dim M$. *If* M *is isometrically immersed in* $\bar{M}$, *then*

$$\bar{d}(M) \geq \frac{1}{\sqrt{\max \bar{K}}} \sin^{-1} \left(\frac{\min \bar{K}}{\max K}\right)^{1/2} . \tag{4.3}$$

(*Here* (4.3) *is nontrivial precisely when* $\bar{M}$ *is a symmetric space of rank* 1.)

Finally we note the following theorem, which is an analogue of Theorem (3.8).

Theorem (4.4). *Suppose that* $\bar{M}$ *is a compact simply connected symmetric space of rank* 1. *Let* M *be a compact Riemannian manifold isometrically immersed in* $\bar{M}$ *with* $\dim \bar{M} < 2 \dim M$. *Write* $n = \dim M$. *Then*

$$\bar{d}(M) \geq \frac{1}{\sqrt{\max \bar{K}}} \cot^{-1} \left(\frac{\max \| H \|}{n\sqrt{\min \bar{K}}}\right).$$

Hence if M *is a minimal variety*, $\bar{d}(M) \geq \frac{\pi}{2} (\max \bar{K})^{-1/2}$.

Bibliography

[1] R. L. Bishop & R. J. Crittenden, *Geometry of manifolds,* Academic Press, New York, 1964.
[2] R. J. Crittenden, *Minimum and conjugate points in symmetric spaces,* Canad. J. Math. **14** (1962) 320–328.
[3] A. Gray, *Minimal varieties and almost Hermitian submanifolds,* Michigan Math. J. **12** (1965) 273–287.
[4] R. Hermann, *Focal points of closed submanifolds of Riemannian spaces,* Nederl. Akad. Wetensch. Proc. Ser. A **66** (1963) 613–628.
[5] J. Milnor, *Lectures on Morse theory,* Ann. Math. Studies No. 54, Princeton University Press, 1963.
[6] B. O'Neill, *Immersion of manifolds of nonpositive curvature,* Proc. Amer. Math. Soc. **11** (1960) 132–134.
[7] ——, *Umbilics of constant curvature immersions,* Duke Math. J. **32** (1965) 149–160.
[8] T. Ôtsuki, *On the existence of solutions of a system of quadratic equations and its geometric applications,* Proc. Japan Acad. **29** (1953) 99–100.
[9] ——, *Isometric imbedding of Riemannian manifolds in a Riemannian manifold,* J. Math. Soc. Japan **6** (1954) 221–234.
[10] E. Stiel, *Immersions into manifolds of constant negative curvature,* Proc. Amer. Math. Soc. **18** (1967) 713–715.

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